J. Math. Anal. Appl. 377 (2011) 303–318

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Semigroupoid C^*-algebras

R. Exel [1]

Departamento de Matemática, Universidade Federal de Santa Catarina, 88040-900, Florianópolis, Brazil

ARTICLE INFO

Article history:
Received 24 June 2010
Available online 28 October 2010
Submitted by Richard M. Aron

Keywords:
Semigroupoid
Higher rank graph
C^*-algebras

ABSTRACT

A semigroupoid is a set equipped with a partially defined associative operation. Given a semigroupoid Λ we construct a C^*-algebra $\mathcal{O}(\Lambda)$ from it. We then present two main examples of semigroupoids, namely the Markov semigroupoid associated to an infinite 0–1 matrix, and the semigroupoid associated to a row-finite higher-rank graph without sources. In both cases the semigroupoid C^*-algebra is shown to be isomorphic to the algebras usually attached to the corresponding combinatorial object, namely the Cuntz–Krieger algebras and the higher-rank graph C^*-algebras, respectively. In the case of a higher-rank graph (Λ, d), it follows that the dimension function d is superfluous for defining the corresponding C^*-algebra.

1. Introduction

The theory of C^*-algebras has greatly benefited from Combinatorics in the sense that some of the most interesting examples of C^*-algebras arise from combinatorial objects, such as the case of graph C^*-algebras [1,2,8,10,12,13,16,21,22,25,26], see also [18] and the references therein. More recently Kumjian and Pask have introduced the notion of *higher-rank graphs* [11], inspired by Robertson and Steger's work on buildings [23,24], which turns out to be another combinatorial object with which an interesting new class of C^*-algebras may be constructed. See also [7,14,15,18,19].

The crucial insight leading to the notion of higher-rank graphs lies in viewing ordinary graphs as *categories* (in which the morphisms are finite paths) equipped with a *length* function possessing a certain unique factorization property (see [11] for more details).

Kumjian and Pask's interesting idea of viewing graphs as categories suggests that one could construct C^*-algebras for more general categories.

Since Eilenberg and Mac Lane introduced the notion of categories in the early 40's, the archetypal idea of composition of functions has been mathematically formulated in terms of categories, whereby a great emphasis is put on the *domain* and *co-domain* of a function. However one may argue that, while the domain is an intrinsic part of a function, co-domains are not so vital. If one imagines a very elementary function f with domain, say $X = \{1, 2\}$, defined by $f(x) = x^2$, one does not really need to worry about its co-domain. But if f is to be seen as a morphism in the category of sets, one needs to first choose a set Y containing the image of f, and only then f becomes an element of $\mathrm{Hom}(X, Y)$. Regardless of the very innocent nature of our function f, it suddenly is made to evoke an enormous amount of morphisms, all of them having the same domain X, but with the wildest possible collection of co-domains.

Addressing this concern one could replace the idea of categories with the following: a big set (or perhaps a class) would represent the collection of all morphisms, regardless of domains, ranges or co-domains. A set of *composable* pairs (f, g) of morphisms would be given in advance and for each such pair one would define a composition fg. Assuming the appropriate associativity axiom one arrives as the notion of a *semigroupoid*, precisely defined in 2.1 below. Should our morphisms be

E-mail address: exel@mtm.ufsc.br.

[1] Partially supported by CNPq.

0022-247X
doi:10.1016/j.jmaa.2010.10.061

actual functions a sensible condition for a pair (f, g) to be composable would be to require the range of g to be contained in the domain of f, but we might also think of more abstract situations in which the morphisms are not necessarily functions.

For example, let $A = \{A(i, j)\}_{i,j\in\mathcal{G}}$, be an infinite 0–1 matrix, where $\mathcal{G}$ is an arbitrary set, and let Λ_A be the set of all finite *admissible words* in $\mathcal{G}$, meaning finite sequences $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, of elements $\alpha_i \in \mathcal{G}$, such that $A(\alpha_i, \alpha_{i+1}) = 1$. Given $\alpha, \beta \in \Lambda_A$ write

$$\alpha = \alpha_1\alpha_2\ldots\alpha_n \quad \text{and} \quad \beta = \beta_1\beta_2\ldots\beta_m,$$

and let us say that α and β are composable if $A(\alpha_n, \beta_1) = 1$, in which case we let $\alpha\beta$ be the concatenated word

$$\alpha\beta = \alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_m.$$

We shall refer to Λ_A as the *Markov* semigroupoid. This category-like structure lacks a notion of *objects* and in fact it cannot always be made into a category. Consider for instance the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we let the index set of A be $\mathcal{G} = \{\alpha_1, \alpha_2\}$, notice that the words α_1 and α_2 may be legally composed to form the words $\alpha_1\alpha_1$, $\alpha_1\alpha_2$, and $\alpha_2\alpha_1$, but $\alpha_2\alpha_2$ is forbidden, precisely because $A(\alpha_2, \alpha_2) = 0$.

Should there exist an underlying category, the fact that, say, $\alpha_1\alpha_2$ is a legal composition would lead one to believe that $s(\alpha_1)$, the domain, or source of α_1 coincides with $r(\alpha_2)$, the co-domain of α_2. But then for similar reasons one would have

$$s(\alpha_2) = r(\alpha_1) = s(\alpha_1) = r(\alpha_2),$$

which would imply that $\alpha_2\alpha_2$ is a valid composition, but it is clearly not. This example was in fact already noticed by Tomforde [25] with the purpose of showing that a 0–1 matrix A is not always the edge matrix of a graph. Although the above matrix may be replaced by another one which is the edge matrix of a graph and gives the same Cuntz–Krieger algebra, the same trick does not work for infinite matrices.

This is perhaps an indication that we should learn to live with semigroupoids which are not true categories. Given the sheer simplicity of the notion of semigroupoid, one can easily fit all of the combinatorial objects so far referred to within the framework of semigroupoids.

The goal of this work is therefore to introduce a notion of *representation* of semigroupoids, with its accompanying universal C^*-algebra, which in turn generalizes earlier constructions such as the Cuntz–Krieger algebras for arbitrary matrices of [6] and the higher-rank graph C^*-algebras of [11], and hence ordinary graph C^*-algebras as well.

The present paper is mostly devoted to comparing semigroupoid C^*-algebras with Cuntz–Krieger and higher-rank graph C^*-algebras. Please see [5], where a deeper study is made of the structure of semigroupoid C^*-algebras, including describing them as groupoid C^*-algebras.

The definition of a representation of a semigroupoid Λ given in 4.1, and consequently of the C^*-algebra of Λ, here denoted $\mathcal{O}(\Lambda)$, is strongly influenced by [6], and hence it is capable of smoothly dealing with the troubles usually caused by *non-row-finiteness*.

Speaking of another phenomenon that requires special attention in graph C^*-algebra theory, the presence of *sources*, once cast in the perspective of semigroupoids, becomes much easier to deal with.

To avoid confusion we use a different term and define a *spring* (rather than source) to be an element f of a semigroupoid Λ for which fg is not a legal multiplication for any $g \in \Lambda$. The sources of graph theory are much the same as our springs, and they cause the same sort of problems, but there are some subtle, albeit important differences. For example, in a category any element f may be right-multiplied by the identity morphism on its domain, and hence categories never have any springs. On the other hand, even though higher-rank graphs are defined as categories, sources may still be present and require a special treatment. See however [5, 18.2.ii].

While springs are irremediably killed when considered within the associated semigroupoid C^*-algebra, as shown in 5.1, it is rather easy to get rid of them by replacing the given semigroupoid by a somewhat canonical spring-less one (3.3). This is specially interesting because a slight correction performed on the ingredient semigroupoid is seen to avoid the need to redesign the whole theoretical apparatus.

As already mentioned, the C^*-algebra $C^*(\Lambda)$ associated to a higher-rank graph (Λ, d) in [11] turns out to be a special case of our construction: since Λ is defined to be a category, it is obviously a semigroupoid, so we may consider its semigroupoid C^*-algebra $\mathcal{O}(\Lambda)$, which we prove to be isomorphic to $C^*(\Lambda)$ in 8.7.

One of the most interesting aspects of this is that the construction of $\mathcal{O}(\Lambda)$ does not use the *dimension function* "*d*" at all, relying exclusively on the algebraic structure of the subjacent category. In other words, this shows that the dimension function is superfluous in the definition of $C^*(\Lambda)$.

It should be stressed that our proof of the isomorphism between $\mathcal{O}(\Lambda)$ and $C^*(\Lambda)$ is done under the standing hypotheses of [11], namely that (Λ, d) is row-finite and has no sources. The reader will not find here a comparison between our construction and the more recent treatment of Farthing, Muhly and Yeend [7] (see also [20]) for general finitely aligned higher rank graphs. We hope to be able to address this issue in a future paper.

It is a consequence of Definition 4.1, describing our notion of a representation S of a semigroupoid Λ, that if Λ contains elements f, g and h such that $fg = fh$, then

$$S_g = S_h.$$

Therefore, even if g and h are different, that difference is blurred when these elements are seen in $\mathcal{O}(\Lambda)$ via the universal representation. This should probably be interpreted as saying that our representation theory is not really well suited to deal with general semigroupoids in which non-monic elements are present. An element f is said to be *monic* if

$$fg = fh \quad \Rightarrow \quad g = h.$$

Fortunately all of our examples consist of semigroupoids containing only monic elements. See Section 4 for more details.

No attempt has been made to consider *topological* semigroupoids although we believe this is a worthwhile program to be pursued. Among a few indications that this can be done is Katsura's topological graphs [9] and Yeend's [27] topological higher-rank graphs, not to mention Renault's pioneering work on groupoids [17].

After recognizing the precise obstruction for interpreting Cuntz–Krieger algebras from the point of view of categories or graphs, one can hardly help but to think of the obvious generalization of higher-rank graphs to semigroupoids based on the unique factorization property. Even though we do not do anything useful based on this concept we spell out the precise definition in 8.1 below. As an example, the Markov semigroupoid for the above 2×2 matrix is a rank 1 semigroupoid which is not a rank 1 graph.

We would also like to mention that although we have not seriously considered the ultra-graph C^*-algebras of Tomforde [25] from a semigroupoid point of view, we believe that these may also be described in terms of naturally occurring semigroupoids.

I would like to acknowledge many fruitful conversations with A. Kumjian, M. Laca and D. Pask during the process of developing this work. Special thanks go to A. Sims for bringing to our attention some important references in the subject of higher-rank graphs.

2. Semigroupoids

In this section we introduce the basic algebraic ingredient of our construction.

Definition 2.1. A semigroupoid is a triple $(\Lambda, \Lambda^{(2)}, \cdot)$ such that Λ is a set, $\Lambda^{(2)}$ is a subset of $\Lambda \times \Lambda$, and

$$\cdot : \Lambda^{(2)} \to \Lambda$$

is an operation which is associative in the following sense: if $f, g, h \in \Lambda$ are such that either

(i) $(f, g) \in \Lambda^{(2)}$ and $(g, h) \in \Lambda^{(2)}$, or
(ii) $(f, g) \in \Lambda^{(2)}$ and $(fg, h) \in \Lambda^{(2)}$, or
(iii) $(g, h) \in \Lambda^{(2)}$ and $(f, gh) \in \Lambda^{(2)}$,

then all of (f, g), (g, h), (fg, h) and (f, gh) lie in $\Lambda^{(2)}$, and

$$(fg)h = f(gh).$$

Moreover, for every $f \in \Lambda$, we will let

$$\Lambda^f = \{g \in \Lambda \colon (f, g) \in \Lambda^{(2)}\}.$$

From now on we fix a semigroupoid Λ.

Definition 2.2. Let $f, g \in \Lambda$. We shall say that f *divides* g, or that g is a *multiple* of f, in symbols $f \mid g$, if either

(i) $f = g$, or
(ii) there exists $h \in \Lambda$ such that $fh = g$.

When $f \mid g$, and $g \mid f$, we shall say that f and g are *equivalent*, in symbols $f \simeq g$.

Perhaps the correct way to write up the above definition is to require that $(f, h) \in \Lambda^{(2)}$ before referring to the product "fh". However we will adopt the convention that, when a statement is made about a freshly introduced element which involves a multiplication, then the statement is implicitly supposed to include the requirement that the multiplication involved is allowed.

Notice that in the absence of anything resembling a unit in Λ, it is conceivable that for some element $f \in \Lambda$ there exists no $u \in \Lambda$ such that $f = fu$. Had we not explicitly included 2.2(i), it would not always be the case that $f \mid f$.

A useful artifice is to introduce a unit for Λ, that is, pick some element in the universe outside Λ, call it 1, and set $\tilde{\Lambda} = \Lambda \dot{\cup} \{1\}$. For every $f \in \Lambda$ put

$$1f = f1 = f.$$

Then, whenever $f \mid g$, regardless of whether $f = g$ or not, there always exists $x \in \tilde{\Lambda}$ such that $g = fx$.

We will find it useful to extend the definition of Λ^f, for $f \in \tilde{\Lambda}$, by putting

$$\Lambda^1 = \Lambda.$$

Nevertheless, even if $f1$ is a meaningful product for every $f \in \Lambda$, we *will not* include 1 in Λ^f.

We should be aware that $\tilde{\Lambda}$ is *not* a semigroupoid. Otherwise, since $f1$ and $1g$ are meaningful products, axiom 2.1(i) would imply that $(f1)g$ is also a meaningful product, but this is clearly not always the case.

It is interesting to understand the extent to which the associativity property fails for $\tilde{\Lambda}$. As already observed, 2.1(i) does fail irremediably when $g = 1$. Nevertheless it is easy to see that 2.1 generalizes to $\tilde{\Lambda}$ in all other cases. This is quite useful, since when we are developing a computation, having arrived at an expression of the form $(fg)h$, and therefore having already checked that all products involved are meaningful, we most often want to proceed by writing

$$\cdots = (fg)h = f(gh).$$

The axiom to be invoked here is 2.1(ii) (or 2.1(iii) in a similar situation), and fortunately not 2.1(i)!

Proposition 2.3. *Division is a reflexive and transitive relation.*

Proof. That division is reflexive follows from the definition. In order to prove transitivity let $f, g \in \Lambda$ be such that $f \mid g$ and $g \mid h$. We must prove that $f \mid h$.

The case in which $f = g$, or $g = h$ is obvious. Otherwise there are u, v in Λ (rather than in $\tilde{\Lambda}$) such that $fu = g$, and $gv = h$. As observed above, it is implicit that $(f, u), (g, v) \in \Lambda^{(2)}$, which implies that

$$(f, u), (fu, v) \in \Lambda^{(2)}.$$

By 2.1(ii) we deduce that $(u, v) \in \Lambda^{(2)}$ and that

$$f(uv) = (fu)v = gv = h,$$

and hence $f \mid h$. □

Division is also invariant under multiplication on the left:

Proposition 2.4. *If $k, f, g \in \Lambda$ are such that $f \mid g$, and $(k, f) \in \Lambda^{(2)}$, then $(k, g) \in \Lambda^{(2)}$ and $kf \mid kg$.*

Proof. The case in which $f = g$ being obvious we assume that there is $u \in \Lambda$ such that $fu = g$. Since $(k, f), (f, u) \in \Lambda^{(2)}$ we conclude from 2.1(i) that (kf, u) and $(k, g) = (k, fu)$ lie in $\Lambda^{(2)}$, and that

$$(kf)u = k(fu) = kg,$$

so $kf \mid kg$. □

The next concept will be crucial to the analysis of the structure of semigroupoids.

Definition 2.5. Let $f, g \in \Lambda$. We shall say that f and g *intersect* if they admit a *common multiple*, that is, an element $m \in \Lambda$ such that $f \mid m$ and $g \mid m$. Otherwise we will say that f and g are *disjoint*. We shall indicate the fact that f and g intersect by writing $f \Cap g$, and when they are disjoint we will write $f \perp g$.

If there exists a right-zero element, that is, an element $0 \in \Lambda$ such that $(f, 0) \in \Lambda^{(2)}$ and $f0 = 0$, for all $f \in \Lambda$, then obviously $f \mid 0$, and hence any two elements intersect. We shall be mostly interested in semigroupoids without a right-zero element.

Employing the unitization $\tilde{\Lambda}$ notice that $f \Cap g$ if and only if there are $x, y \in \tilde{\Lambda}$ such that $fx = gy$.

The last important concept, borrowed from the Theory of Categories, is as follows:

Definition 2.6. We shall say that an element $f \in \Lambda$ is *monic* if for every $g, h \in \Lambda$ we have

$$fg = fh \quad \Rightarrow \quad g = h.$$

3. Springs

We would now like to discuss certain special properties of elements $f \in \Lambda$ for which $\Lambda^f = \emptyset$. It would be sensible to call these elements *sources*, following the terminology adopted in Graph Theory, but given some subtle differences we'd rather use another term:

Definition 3.1. We will say that an element f of a semigroupoid Λ is a *spring* when $\Lambda^f = \emptyset$.

Springs are sometimes annoying, so we shall now discuss a way of getting rid of springs. Let us therefore fix a semigroupoid Λ which has springs.

Denote by Λ_0 the subset of Λ formed by all springs and let E' be a set containing a distinct element e'_g, for every $g \in \Lambda_0$. Consider any equivalence relation "$\sim$" on E' according to which

$$e'_g \sim e'_{fg}, \tag{3.2}$$

for any spring g, and any f such that $g \in \Lambda^f$. Observe that fg is necessarily also a spring since $\Lambda^{fg} = \Lambda^g$, by 2.1(i)–(ii). For example, one can take the equivalence relation according to which any two elements are related. Alternatively we could use the smallest equivalence relation satisfying (3.2).

We shall denote the quotient space $E'/\sim$ by E, and for every spring g we will denote the equivalence class of e'_g by e_g. Unlike the e'_g, the e_g are obviously no longer distinct elements. In particular we have

$$e_g = e_{fg}, \quad \forall f \in \Lambda,\ \forall g \in \Lambda_0 \cap \Lambda^f.$$

We shall now construct a semigroupoid Γ as follows: set $\Gamma = \Lambda \mathbin{\dot\cup} E$, and put

$$\Gamma^{(2)} = \Lambda^{(2)} \cup \big\{(g, e_g)\colon\ g \in \Lambda_0\big\} \cup \big\{(e_g, e_g)\colon\ g \in \Lambda_0\big\}.$$

Define the multiplication

$$\cdot : \Gamma^{(2)} \to \Gamma,$$

to coincide with the multiplication of Λ when restricted to $\Lambda^{(2)}$, and moreover set

$$g \cdot e_g = g \quad \text{and} \quad e_g \cdot e_g = e_g, \quad \forall g \in \Lambda_0.$$

It is rather tedious, but entirely elementary, to show that Γ is a semigroupoid without any springs containing Λ. To summarize the conclusions of this section we state the following:

Theorem 3.3. *For any semigroupoid Λ there exists a spring-less semigroupoid Γ containing Λ.*

Given a certain freedom in the choice of the equivalence relation "$\sim$" above, there seems not to be a canonical way to embed Λ in a spring-less semigroupoid. The user might therefore have to make a case by case choice according to his or her preference.

4. Representations of semigroupoids

In this section we begin the study of the central notion bridging semigroupoids and operator algebras.

Definition 4.1. Let Λ be a semigroupoid and let B be a unital C^*-algebra. A mapping $S : \Lambda \to B$ will be called a *representation of Λ in B*, if for every $f, g \in \Lambda$, one has that:

(i) S_f is a partial isometry,
(ii) $S_f S_g = \begin{cases} S_{fg}, & \text{if } (f,g) \in \Lambda^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$

Moreover the initial projections $Q_f = S_f^* S_f$, and the final projections $P_g = S_g S_g^*$, are required to commute amongst themselves and to satisfy

(iii) $P_f P_g = 0$, if $f \perp g$,
(iv) $Q_f P_g = P_g$, if $(f,g) \in \Lambda^{(2)}$.

Notice that if $(f,g)\notin\Lambda^{(2)}$, then $Q_fP_g = S_f^*S_fS_gS_g^* = 0$, by (ii). Complementing (iv) above we could therefore add:

(v) $Q_fP_g = 0$, if $(f,g)\notin\Lambda^{(2)}$.

We will automatically extend any representation S to the unitization $\tilde{\Lambda}$ by setting $S_1 = 1$. Likewise we put $Q_1 = P_1 = 1$.

Notice that in case Λ contains an element f which is not monic, say $fg = fh$, for a pair of distinct elements $g, h\in\Lambda$, one necessarily has $S_g = S_h$, for every representation S. In fact

$$S_g = S_gS_g^*S_g = P_gS_g = Q_fP_gS_g = S_f^*S_fS_gS_g^*S_g = S_f^*S_fS_g = S_f^*S_{fg},$$

and similarly $S_h = S_f^*S_{fh}$, so it follows that $S_g = S_h$, as claimed.

This should probably be interpreted as saying that our representation theory is not really well suited to deal with general semigroupoids in which non-monic elements are present. In fact, all of our examples consist of semigroupoids containing only monic elements.

From now on we will fix a representation S of a given semigroupoid Λ in a unital C^*-algebra B. By 4.1(iv) we have that $P_h \leqslant Q_f$, for all $h\in\Lambda^f$, so if $h_1, h_2\in\Lambda^f$ we deduce that

$$P_{h_1}\vee P_{h_2} := P_{h_1} + P_{h_2} - P_{h_1}P_{h_2} \leqslant Q_f.$$

More generally, if H is a finite subset of Λ^f we will have

$$\bigvee_{h\in H} P_h \leqslant Q_f.$$

We now wish to discuss whether or not the above inequality becomes an identity under circumstances which we now make explicit:

Definition 4.2. Let X be any subset of Λ. A subset $H\subseteq X$ will be called a *covering* of X if for every $f\in X$ there exists $h\in H$ such that $h \Cap f$. If moreover the elements of H are mutually disjoint then H will be called a *partition* of X.

The following elementary fact is noted for further reference:

Proposition 4.3. *A subset $H\subseteq X$ is a partition of X if and only if H is a maximal subset of X consisting of pairwise disjoint elements.*

Returning to our discussion above we wish to require that

$$\bigvee_{h\in H} P_h \stackrel{?}{=} Q_f, \tag{4.4}$$

whenever H is a covering of Λ^f. The trouble with this equation is that when H is infinite there is no reasonable topology available on B under which one can make sense of the supremum of infinitely many commuting projections.

Before we try to attach any sense to (4.4) notice that if $g\in\tilde{\Lambda}$ and $h\in\Lambda\setminus\Lambda^g$, then $P_h\leqslant 1 - Q_g$, by 4.1(v), and hence also

$$\bigvee_{h\in H} P_h \leqslant 1 - Q_g,$$

for every finite set $H\subseteq\Lambda\setminus\Lambda^g$. More generally, given finite subsets $F, G\subseteq\tilde{\Lambda}$, denote

$$\Lambda^{F,G} = \Big(\bigcap_{f\in F}\Lambda^f\Big)\cap\Big(\bigcap_{g\in G}\Lambda\setminus\Lambda^g\Big),$$

and let $h\in\Lambda^{F,G}$. By 4.1(iv)–(v), we have that

$$P_h \leqslant \prod_{f\in F} Q_f \prod_{g\in G}(1 - Q_g).$$

As in the above cases we deduce that

$$\bigvee_{h\in H} P_h \leqslant \prod_{f\in F} Q_f \prod_{g\in G}(1 - Q_g),$$

for every finite subset $H\subseteq\Lambda^{F,G}$.

Definition 4.5. A representation S of Λ in a unital C^*-algebra B is said to be *tight* if for every pair of finite subsets $F, G \subseteq \tilde{\Lambda}$, and for every finite covering H of $\Lambda^{F,G}$ one has that

$$\bigvee_{h\in H} P_h = \prod_{f\in F} Q_f \prod_{g\in G}(1 - Q_g).$$

Observe that if for every pair of finite subsets $F, G \subseteq \tilde{\Lambda}$, one has that $\Lambda^{F,G}$ admits no finite covering, then any representation is tight by default.

For many representation theories there is a C^*-algebra whose representations are in one-to-one correspondence with the representations in the given theory. Semigroupoid representations are no exception:

Definition 4.6. Given a semigroupoid Λ we shall let $\tilde{\mathcal{O}}(\Lambda)$ be the universal unital C^*-algebra generated by a family of partial isometries $\{S_f\}_{f\in\Lambda}$ subject to the relations that the correspondence $f \mapsto S_f$ is a tight representation of Λ. That representation will be called the *universal representation* and the closed $*$-subalgebra of $\tilde{\mathcal{O}}(\Lambda)$ generated by its range will be denoted $\mathcal{O}(\Lambda)$.

It is clear that $\tilde{\mathcal{O}}(\Lambda)$ is either equal to $\mathcal{O}(\Lambda)$ or to its unitization. Observe also that the relations we are referring to in the above definition are all expressable in the form described in [3]. Moreover these relations are admissible, since any partial isometry has norm one. It therefore follows that $\tilde{\mathcal{O}}(\Lambda)$ exists.

The universal property of $\tilde{\mathcal{O}}(\Lambda)$ may be expressed as follows:

Proposition 4.7. *For every tight representation T of Λ in a unital C^*-algebra B there exists a unique $*$-homomorphism*

$$\varphi : \tilde{\mathcal{O}}(\Lambda) \to B,$$

such that $\varphi(S_f) = T_f$, for every $f \in \Lambda$.

It might also be interesting to define a "Toeplitz" extension of $\tilde{\mathcal{O}}(\Lambda)$, as the universal unital C^*-algebra generated by a family of partial isometries $\{S_f\}_{f\in\Lambda}$ subject to the relations that the correspondence $f \mapsto S_f$ is a (not necessarily tight) representation of Λ. If such an algebra is denoted $\mathcal{T}(\Lambda)$, it is immediate that $\tilde{\mathcal{O}}(\Lambda)$ is a quotient of $\mathcal{T}(\Lambda)$.

As already observed the usefulness of these constructions is probably limited to the case in which every element of Λ is monic.

5. Tight representations and springs

Tight representations and springs do not go together well, as explained below:

Proposition 5.1. *Let S be a tight representation of a semigroupoid Λ and let $f \in \Lambda$ be a spring* (*as defined in* 3.1)*. Then $S_f = 0$.*

Proof. Under the assumption that $\Lambda^f = \emptyset$, notice that the empty set is a covering of Λ^f and hence $Q_f = 0$, by 4.5. Since $Q_f = S_f^* S_f$, one has that $S_f = 0$, as well. □

We thus see that springs do not play any role with respect to tight representations. There are in fact some other non-spring elements on which every tight representation vanishes. Consider for instance the situation in which Λ^f consists of a finite number of elements, say $\Lambda^f = \{h_1, \dots, h_n\}$, each h_i being a spring. Then Λ^f is a finite cover of itself and hence by 4.5 we have

$$Q_f = \bigvee_{i=1}^{n} P_{h_i} = 0,$$

which clearly implies that $S_f = 0$.

One might feel tempted to redesign the whole concept of tight representations especially if one is bothered by the fact that springs are killed by them. However we strongly feel that the right thing to do is to redesign the semigroupoid instead, using 3.3 to replace Λ by a spring-less semigroupoid containing it.

In this case it might be useful to understand the following situation:

Proposition 5.2. *Let S be a tight representation of a semigroupoid Λ and suppose that $f \in \Lambda$ is such that Λ^f contains a single element e such that $e^2 = e$. Then S_e is a projection and moreover $S_e = Q_f$.*

Proof. Since $e^2 = e$, we have that $S_e^2 = S_e$. But since S_e is also a partial isometry, it must necessarily be a projection. By assumption we have that $\{e\}$ is a finite covering for Λ^f so

$$Q_f = P_e = S_e S_e^* = S_e. \qquad \square$$

With this in mind we will occasionally work under the assumption that our semigroupoid has no springs.

6. The Markov semigroupoid

In this section we shall present a semigroupoid whose C^*-algebra is isomorphic to the Cuntz–Krieger algebra introduced in [6]. For this let $\mathcal{G}$ be any set and let $A = \{A(i, j)\}_{i,j\in\mathcal{G}}$ be an arbitrary matrix with entries in $\{0, 1\}$. We consider the set $\Lambda = \Lambda_A$ of all finite *admissible* words

$$\alpha = \alpha_1\alpha_2 \ldots \alpha_n,$$

i.e., finite sequences of elements $\alpha_i \in \mathcal{G}$, such that $A(\alpha_i, \alpha_{i+1}) = 1$. Even though it is sometimes interesting to consider the empty word as valid, we shall *not* do so. If allowed, the empty word would duplicate the role of the extra element $1 \in \tilde{\Lambda}$. Our words are therefore assumed to have strictly positive length ($n \geqslant 1$).

Given another admissible word, say $\beta = \beta_1\beta_2 \ldots \beta_m$, the concatenated word

$$\alpha\beta := \alpha_1 \ldots \alpha_n\beta_1 \ldots \beta_m$$

is admissible as long as $A(\alpha_n, \beta_1) = 1$. Thus, if we set

$$\Lambda^{(2)} = \big\{(\alpha, \beta) = (\alpha_1\alpha_2 \ldots \alpha_n, \beta_1\beta_2 \ldots \beta_m) \in \Lambda \times \Lambda\colon\ A(\alpha_n, \beta_1) = 1\big\},$$

we get a semigroupoid with concatenation as product.

Definition 6.1. The semigroupoid $\Lambda = \Lambda_A$ defined above will be called the *Markov* semigroupoid.

Observe that the springs in Λ are precisely the words $\alpha = (\alpha_1\alpha_2 \ldots \alpha_n)$ for which $A(\alpha_n, j) = 0$, for every $j \in \mathcal{G}$, that is, for which the α_nth row of A is zero. To avoid springs we will assume that no row of A is zero.

Theorem 6.2. *Suppose that A has no zero rows. Then $\tilde{\mathcal{O}}(\Lambda)$ is $*$-isomorphic to the Exel–Laca algebra $\tilde{\mathcal{O}}_A$* [6, 7.1].

Proof. Throughout this proof we will denote the standard generators of $\tilde{\mathcal{O}}_A$ by $\{\check{S}_x\}_{x\in\mathcal{G}}$, together with their initial and final projections $\check{Q}_x = \check{S}_x^*\check{S}_x$ and $\check{P}_x = \check{S}_x\check{S}_x^*$, respectively. Likewise the standard generators of $\tilde{\mathcal{O}}(\Lambda)$ will be denoted by $\{\hat{S}_f\}_{f\in\Lambda}$, along with their initial and final projections $\hat{Q}_f = \hat{S}_f^*\hat{S}_f$ and $\hat{P}_f = \hat{S}_f\hat{S}_f^*$. In addition, for every $x \in \mathcal{G}$ we will identify the one-letter word "x" with the element x itself, so we may think of $\mathcal{G}$ as a subset of Λ.

We begin by claiming that the set of partial isometries

$$\{\hat{S}_x\}_{x\in\mathcal{G}} \subseteq \tilde{\mathcal{O}}(\Lambda)$$

satisfies the defining relations of $\tilde{\mathcal{O}}_A$, namely TCK_1, TCK_2, and TCK_3 of [6, Section 3], plus [6, 1.3].

Conditions TCK_1 and TCK_2 follow immediately from 4.1, and the observation that if x and y are distinct elements of $\mathcal{G}$, then $x \perp y$ as elements of Λ.

When $A(i, j) = 1$ we have that $(i, j) \in \Lambda^{(2)}$ and hence $\hat{P}_i\hat{Q}_j = \hat{P}_j = A(i, j)\hat{P}_j$, by 4.1(iv). Otherwise, if $A(i, j) = 0$, we have that $(i, j) \notin \Lambda^{(2)}$ and hence $\hat{P}_i\hat{Q}_j = 0 = A(i, j)\hat{P}_j$, by 4.1(v). This proves TCK_3.

In order to prove [6, 1.3] let X, Y be finite subsets of $\mathcal{G}$ such that

$$A(X, Y, j) := \prod_{x\in X} A(x, j) \prod_{y\in Y} \big(1 - A(y, j)\big) \tag{6.2.1}$$

equals zero for all but finitely many j's. It is then easy to see that

$$Z := \big\{j \in \mathcal{G}\colon\ A(X, Y, j) \neq 0\big\}$$

is a finite partition of $\Lambda^{X,Y}$, so

$$\prod_{x\in X} \hat{Q}_x \prod_{y\in Y} (1 - \hat{Q}_y) = \bigvee_{j\in Z} \hat{P}_j = \sum_{j\in Z} \hat{P}_j = \sum_{j\in\mathcal{G}} A(X, Y, j)\hat{P}_j,$$

because the canonical representation $f \in \Lambda \mapsto \hat{S}_f \in \tilde{\mathcal{O}}(\Lambda)$ is tight by definition. It then follows from the universal property of $\tilde{\mathcal{O}}_A$ that there exists a $*$-homomorphism

$$\Phi : \tilde{\mathcal{O}}_A \to \tilde{\mathcal{O}}(\Lambda), \tag{6.2.2}$$

such that $\Phi(\check{S}_x) = \hat{S}_x$, for every $x \in \mathcal{G}$.

Next consider the map $\check{S} : \Lambda \to \tilde{\mathcal{O}}_A$ defined as follows: given $\alpha \in \Lambda$, write $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$, with $\alpha_i \in \mathcal{G}$, and put

$$\check{S}_\alpha = \check{S}_{\alpha_1}\check{S}_{\alpha_2} \ldots \check{S}_{\alpha_n}.$$

We claim that $\check{S}$ is a tight representation of Λ in $\tilde{\mathcal{O}}_A$. The first two axioms of 4.1 are immediate, while the commutativity of the $\check{P}_f$, and $\check{Q}_g$ follow from [6, 3.2] and [4, 2.4.iii]. Next suppose that $\alpha, \beta \in \Lambda$ are such that $\alpha \perp \beta$. One may then prove that

$$\alpha = \alpha_1 \ldots \alpha_p \ldots \alpha_n \quad \text{and} \quad \beta = \beta_1 \ldots \beta_p \ldots \beta_m,$$

with $1 \leqslant p \leqslant n, m$, and such that $\alpha_i = \beta_i$ for $i < p$, and $\alpha_p \neq \beta_p$. Denoting by $\gamma = \alpha_1 \ldots \alpha_{p-1}$ (possibly the empty word), we have that

$$\check{P}_\alpha \leqslant \check{S}_\gamma \check{S}_{\alpha_p} \check{S}^*_{\alpha_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\alpha_p} \check{S}^*_\gamma,$$

and similarly $\check{P}_\beta \leqslant \check{S}_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma$. It follows that

$$\check{P}_\alpha \check{P}_\beta \leqslant \check{S}_\gamma \check{P}_{\alpha_p} \check{S}^*_\gamma \check{S}_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\alpha_p} \check{Q}_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\alpha_p} \check{P}_{\beta_p} \check{Q}_\gamma \check{S}^*_\gamma = 0,$$

by [6, TCK$_2$], hence proving 4.1(iii). In order to verify 4.1(iv) let $(\alpha, \beta) \in \Lambda^{(2)}$, so that $A(\alpha_n, \beta_1) = 1$, where n is the length of α. As shown in "Claim 1" in the proof of [6, 3.2], we have that $\check{Q}_\alpha = \check{Q}_{\alpha_n}$, so

$$\check{Q}_\alpha \check{P}_\beta = \check{Q}_{\alpha_n} \check{P}_{\beta_1} \check{P}_\beta = \check{P}_{\beta_1} \check{P}_\beta = \check{P}_\beta,$$

where we have used TCK$_3$ in the second equality.

We are then left with the task of proving $\check{S}$ to be tight. For this let X and Y be finite subsets of Λ and let Z be a finite covering of $\Lambda^{X,Y}$. We must prove that

$$\bigvee_{h \in Z} \check{P}_h = \prod_{f \in X} \check{Q}_f \prod_{g \in Y} (1 - \check{Q}_g). \tag{6.2.3}$$

Using TCK$_3$ it is easy to check the inequality "$\leqslant$" in (6.2.3) so it suffices to verify the opposite inequality.

Let $h_1, h_2 \in Z$ be such that $h_1 \Cap h_2$, and write $h_1x_1 = h_2x_2$, where $x_1, x_2 \in \tilde{\Lambda}$. Assuming that the length of h_1 does not exceed that of h_2, one sees that h_1 is an initial segment of h_2, and hence $h_1 \mid h_2$. Any element of $\Lambda^{X,Y}$ which intersects h_2 must therefore also intersect h_1. This said we see that $Z' := Z \setminus \{h_2\}$ is also a covering of $\Lambda^{X,Y}$. Since the left-hand side of (6.2.3) decreases upon replacing Z by Z', it is clearly enough to prove the remaining inequality "$\geqslant$" with Z' in place of Z.

Proceeding in such a way every time we find pairs of intersecting elements in Z we may then suppose that Z consists of pairwise disjoint elements, and hence that Z is a partition.

Given $f \in \Lambda$, write $f = \alpha_1 \ldots \alpha_n$, with $\alpha_i \in \mathcal{G}$, and observe that $\check{Q}_f = \check{Q}_{\alpha_n}$, as already mentioned. Since $\Lambda^f = \Lambda^{\alpha_n}$, as well, we may assume without loss of generality that X and Y consist of words of length one, or equivalently that $X, Y \subseteq \mathcal{G}$. Let

$$J = \{j \in \mathcal{G}\colon A(X, Y, j) \neq 0\},$$

where $A(X, Y, j)$ is as in (6.2.1). Notice that $j \in J$ if and only if $A(x, j) = 1$, and $A(y, j) = 0$, for all $x \in X$ and $y \in Y$, which is precisely to say that $j \in \Lambda^{X,Y}$. In other words

$$J = \Lambda^{X,Y} \cap \mathcal{G}.$$

It is clear that J shares with Z the property of being maximal among the subsets of pairwise disjoint elements of $\Lambda^{X,Y}$ (see 4.3).

Suppose for the moment that Z is formed by words of length one, i.e., that $Z \subseteq \mathcal{G}$. Then $Z \subseteq J$, and so $Z = J$, by maximality. This implies that J is finite and

$$\bigvee_{z \in Z} \check{P}_z = \bigvee_{j \in J} \check{P}_j = \sum_{j \in J} \check{P}_j = \sum_{j \in \mathcal{G}} A(X, Y, j)\check{P}_j = \prod_{x \in X} \check{Q}_x \prod_{y \in Y} (1 - \check{Q}_y),$$

by [6, 1.3], thus proving (6.2.3). Addressing the situation in which Z is not necessarily contained in $\mathcal{G}$, let

$$Z_j = \{\alpha \in Z\colon \alpha_1 = j\}, \quad \forall j \in J.$$

Since $Z \subseteq \Lambda^{X,Y}$ it is evident that

$$Z = \bigcup_{j \in J} Z_j.$$

Moreover notice that each Z_j is nonempty since otherwise $Z \cup \{j\}$ will be a subset of $\Lambda^{X,Y}$ formed by mutually disjoint elements, contradicting the maximality of Z. In particular this shows that J is finite and hence we may use [6, 1.3], so that

$$\prod_{x \in X} \check{Q}_x \prod_{y \in Y} (1 - \check{Q}_y) = \sum_{j \in \mathcal{G}} A(X, Y, j) \check{P}_j = \sum_{j \in J} \check{P}_j. \tag{6.2.4}$$

We claim that for every $j \in J$ one has that

$$\check{P}_j = \sum_{z \in Z_j} \check{P}_z.$$

Before proving the claim let us notice that it does imply our goal, for then

$$\sum_{z \in Z} \check{P}_z = \sum_{j \in J} \sum_{z \in Z_j} \check{P}_z = \sum_{j \in J} \check{P}_j \overset{(6.2.4)}{=} \prod_{x \in X} \check{Q}_x \prod_{y \in Y} (1 - \check{Q}_y),$$

proving (6.2.3).

Noticing that each Z_j is maximal among subsets of mutually disjoint elements beginning in j, the claim follows from the following:

Lemma 6.3. *Given $x \in \mathcal{G}$, let $\Lambda(x) = \{\alpha \in \Lambda\colon \alpha_1 = x\}$, and let H be a finite partition of $\Lambda(x)$. Then*

$$\sum_{h \in H} \check{P}_h = \check{P}_x.$$

Proof. Let n be the maximum length of the elements of H. We will prove the statement by induction on n. If $n = 1$ it is clear that $H = \{x\}$ and the conclusion follows by obvious reasons. Supposing that $n > 1$ observe that $x \notin H$, or else any element in H with length n will intersect x, violating the hypothesis that H consists of mutually disjoint elements. Therefore every element of H has length at least two.

Let $J = \{j \in \mathcal{G}\colon A(x, j) = 1\}$ and set $H_j = \{\alpha \in H\colon \alpha_2 = j\}$. It is clear that

$$H = \bigcup_{j \in J} H_j.$$

Moreover notice that every H_j is nonempty, since otherwise $H \cup \{xj\}$ consists of mutually disjoint elements and properly contains H, contradicting maximality. In particular this implies that J is finite and hence by [6, 1.3] we have

$$\check{Q}_x = \sum_{j \in \mathcal{G}} A(x, j) \check{P}_j = \sum_{j \in J} \check{P}_j. \tag{6.3.1}$$

For every $j \in J$, let H'_j be the set obtained by deleting the first letter from all words in H_j, so that $H'_j \subseteq \Lambda(j)$, and $H_j = xH'_j$. One moment of reflexion will convince the reader that H'_j is maximal among the subsets of mutually disjoint elements of $\Lambda(j)$. Since the maximum length of elements in H'_j is no bigger than $n - 1$, we may use induction to conclude that

$$\check{P}_j = \sum_{k \in H'_j} \check{P}_k.$$

Therefore

$$P_x = \check{S}_x \check{S}_x^* \check{S}_x \check{S}_x^* = \check{S}_x \check{Q}_x \check{S}_x^* \overset{(6.3.1)}{=} \sum_{j \in J} \check{S}_x \check{P}_j \check{S}_x^* = \sum_{j \in J} \sum_{k \in H'_j} \check{S}_x \check{P}_k \check{S}_x^* = \sum_{j \in J} \sum_{k \in H'_j} \check{P}_{xk} = \sum_{j \in J} \sum_{h \in H_j} \check{P}_h = \sum_{h \in H} \check{P}_h. \qquad \square$$

Returning to the proof of 6.2, now in possession of the information that $\check{S}$ is a tight representation of Λ, we conclude by the universal property of $\tilde{\mathcal{O}}(\Lambda)$ that there exists a $*$-homomorphism

$$\Psi : \tilde{\mathcal{O}}(\Lambda) \to \tilde{\mathcal{O}}_A,$$

such that $\Psi(\hat{S}_\alpha) = \check{S}_\alpha$, for all $\alpha \in \Lambda$. It is then clear that Ψ is the inverse of the homomorphism Φ of (6.2.2), and hence both Φ and Ψ are isomorphisms. $\square$

7. Categories

In this section we fix a small category Λ. Notice that the collection of all morphisms of Λ (which we identify with Λ itself) is a semigroupoid under composition. We shall now study Λ from the point of view of the theory introduced in the previous sections.

Given $v \in \mathrm{obj}(\Lambda)$ (meaning the set of objects of Λ) we will identify v with the identity morphism on v, so that we will see $\mathrm{obj}(\Lambda)$ as a subset of the set of all morphisms.

Given $f \in \Lambda$ we will denote by $s(f)$ and $r(f)$ the *domain* and *co-domain* of f, respectively. Thus the set of all composable pairs may be described as

$$\Lambda^{(2)} = \big\{(f,g) \in \Lambda \times \Lambda \colon s(f) = r(g)\big\}.$$

Given $f \in \Lambda$ notice that $\Lambda^f = \{g \in \Lambda\colon s(f) = r(g)\}$. In particular, if $v \in \mathrm{obj}(\Lambda)$ then $s(v) = r(v) = v$, so

$$\Lambda^v = \big\{g \in \Lambda\colon r(g) = v\big\}.$$

A category is a special sort of semigroupoid in several ways. For example, if f_1, f_2, g_1 and $g_2 \in \Lambda$ are such that $(f_i, g_i) \in \Lambda^{(2)}$ for all i, j, except perhaps for $(i, j) = (2, 2)$, then necessarily $(f_2, g_2) \in \Lambda^{(2)}$, because

$$s(f_2) = r(g_1) = s(f_1) = r(g_2).$$

Another special property of a category among semigroupoids is the fact that for every $f \in \Lambda$ there exists $u \in \Lambda^f$ such that $f = fu$, namely one may take u to be (the identity on) $s(f)$. Thus $f \mid f$ even if we had omitted 2.2(i) in the definition of division. Clearly this also implies that Λ has no springs.

From now on we fix a representation S of Λ in a unital C^*-algebra B and denote by Q_f and P_f, the initial and final projections of each S_f, respectively. A few elementary facts are in order:

Proposition 7.1.

(i) *For every $v \in \mathrm{obj}(\Lambda)$ one has that S_v is a projection, and hence $S_v = P_v = Q_v$.*
(ii) *If u and v are distinct objects then $P_u \perp P_v$.*
(iii) *For every $f \in \Lambda$ one has that $Q_f = P_{s(f)}$.*

Proof. We leave the elementary proof of (i) to the reader. Given distinct objects u and v it is clear that $u \perp v$, so $P_u \perp P_v$, by 4.1(iii). With respect to (iii) we have

$$Q_f = S_f^* S_f = S_f^* S_{fs(f)} = S_f^* S_f S_{s(f)} = Q_f P_{s(f)} = P_{s(f)},$$

where the last equality follows from 4.1(iv). □

Definition 7.2. Let H be a Hilbert space and let $S : \Lambda \to \mathcal{B}(H)$ be a representation. We will say that Λ is *nondegenerate* if the closed $*$-subalgebra of $\mathcal{B}(H)$ generated by the range of S is nondegenerate.

Nondegenerate Hilbert space representations are partly tight in the following sense:

Proposition 7.3. *Let $S : \Lambda \to \mathcal{B}(H)$ be a representation. If either*

(i) *S is nondegenerate, or*
(ii) $\mathrm{obj}(\Lambda)$ *is infinite,*

then for every pair of finite subsets $F, G \subseteq \Lambda$ such that $\Lambda^{F,G} = \emptyset$, one has that

$$\prod_{f\in F} Q_f \prod_{g\in G} (1 - Q_g) = 0.$$

Proof. Notice that $\Lambda^{F,G} = \emptyset$ implies that

$$\left(\bigcap_{f\in F} \Lambda^f\right) \subseteq \Lambda \setminus \left(\bigcap_{g\in G} \Lambda \setminus \Lambda^g\right) = \bigcup_{g\in G} \Lambda^g. \tag{7.3.1}$$

Case 1. Assuming that $F \neq \emptyset$, let $f_0 \in F$. Then either there is some $f \in F$, with $s(f) \neq s(f_0)$, in which case

$$Q_{f_0} Q_f = P_{s(f_0)} P_{s(f)} = 0,$$

proving the statement; or $s(f) = s(f_0)$, for all $f \in F$. Therefore we may suppose that $s(f_0)$ belongs to Λ^f for every $f \in F$, and hence by (7.3.1) there exists $g_0 \in G$ such that $s(f_0) \in \Lambda^{g_0}$. But this is only possible if $s(f_0) = s(g_0)$ and hence

$$Q_{f_0}(1 - Q_{g_0}) = P_{s(f_0)}(1 - P_{s(g_0)}) = 0,$$

concluding the proof in Case 1.

Case 2. Assuming next that $F = \emptyset$, we claim that

$$\mathrm{obj}(\Lambda) = \{s(g)\colon\ g \in G\}.$$

In fact, arguing as in (7.3.1) one has that $\bigcup_{g \in G} \Lambda^g = \Lambda$, so for every $v \in \mathrm{obj}(\Lambda)$ there exists g in G such that $v \in \Lambda^g$, whence $v = s(g)$, proving our claim.

Under the assumption that $\mathrm{obj}(\Lambda)$ is infinite we have reached a contradiction, meaning that Case 2 is impossible and the proof is concluded. We thus proceed supposing nondegeneracy. Let

$$R = \prod_{g \in G} (1 - Q_g),$$

so, proving the statement is equivalent to proving that $R = 0$. Given $v \in \mathrm{obj}(\Lambda)$, let $g \in G$ be such that $v = s(g)$. Then

$$Q_g S_v = P_{s(g)} S_v = S_v,$$

from where we deduce that

$$R S_v = R(1 - Q_g) S_v = 0.$$

Given any $f \in \Lambda$ we then have that

$$R S_f = R S_{r(f)} S_f = 0 \quad \text{and} \quad R S_f^* = R S_{s(f)} S_f^* = 0,$$

so $R = 0$, by nondegeneracy. □

We next present a greatly simplified way to check that a representation of Λ is tight.

Proposition 7.4. *Given a representation $S : \Lambda \to \mathcal{B}(H)$, consider the following two statements*:

(a) *S is tight.*
(b) *For every $v \in \mathrm{obj}(\Lambda)$ and every finite covering H of Λ^v one has that $\bigvee_{h \in H} P_h = P_v$.*

Then

(i) (a) *implies* (b).
(ii) *If S is nondegenerate, or* $\mathrm{obj}(\Lambda)$ *is infinite, then* (b) *implies* (a).

Proof. (i) Assume that S is tight and that H is a finite covering of Λ^v. Setting $F = \{v\}$ and $G = \emptyset$, notice that

$$\Lambda^{F,G} = \Lambda^v,$$

so H is a finite covering of $\Lambda^{F,G}$, and hence we have by definition that

$$\bigvee_{h \in H} P_h = \prod_{f \in F} Q_f \prod_{g \in G} (1 - Q_g) = Q_v = P_v.$$

(ii) Assuming S nondegenerate, or $\mathrm{obj}(\Lambda)$ infinite, we next prove that (b) implies (a). So let F and G be finite subsets of Λ and let H be a finite covering of $\Lambda^{F,G}$. We must prove that the identity in 4.5 holds. If $\Lambda^{F,G} = \emptyset$, the conclusion follows from 7.3. So we assume that $\Lambda^{F,G} \neq \emptyset$.

Case 1. $F \neq \emptyset$. Pick $h \in \Lambda^{F,G}$ and notice that for every $f \in F$ one has that $s(f) = r(h)$, and for every $g \in G$, it is the case that $s(g) \neq r(h)$. It therefore follows that

$$\Lambda^{F,G} = \Lambda^v,$$

where $v = r(h)$, so H is in fact a covering of Λ^v. By hypothesis we then have that

$$\bigvee_{h \in H} P_h = P_v. \tag{7.4.1}$$

On the other hand observe that for every $g \in G$, we have that

$$Q_g \overset{7.1(\mathrm{iii})}{=} P_{s(g)} \perp P_v,$$

given that $s(g) \neq v$. Noticing that for $f \in F$, we have $Q_f = P_{s(f)} = P_v$, we deduce that

$$\prod_{f \in F} Q_f \prod_{g \in G} (1 - Q_g) = P_v \overset{(7.4.1)}{=} \bigvee_{h \in H} P_h,$$

proving that the identity in 4.5 indeed holds in case $F \neq \emptyset$.

Case 2. $F = \emptyset$. Let

$$V = \mathrm{obj}(\Lambda) \setminus \{s(g)\colon g \in G\},$$

so that

$$\Lambda^{F,G} = \bigcup_{v \in V} \Lambda^v.$$

Given that H is a finite covering of $\Lambda^{F,G}$, we have that for each $v \in V$ there exists $h \in H$ such that $v \Cap h$, which in turn implies that $r(h) = v$. Therefore V is finite and hence so is $\mathrm{obj}(\Lambda)$.

Thus, Case 2 is impossible under the hypothesis that $\mathrm{obj}(\Lambda)$ is infinite, and hence the proof is finished under that hypothesis. We therefore proceed supposing nondegeneracy. It is then easy to show that

$$\sum_{v \in \mathrm{obj}(\Lambda)} P_v = 1,$$

and hence

$$\prod_{g \in G} (1 - Q_g) \overset{7.1(\mathrm{iii})}{=} \prod_{g \in G} (1 - P_{s(g)}) = \sum_{v \in V} P_v. \tag{7.4.2}$$

By assumption H is contained in $\Lambda^{F,G}$, and hence the range of each $h \in H$ belongs to V. Thus

$$H = \dot{\bigcup_{v \in V}} H_v,$$

where $H_v = \{h \in H\colon r(h) = v\}$. Observe that H_v is a covering for Λ^v, since if $f \in \Lambda^v$, there exists some $h \in H$ with $h \Cap f$, but this implies that $r(h) = r(f) = v$, and hence $h \in H_v$. Thus

$$\bigvee_{h \in H} P_h = \bigvee_{v \in V} \left(\bigvee_{h \in H_v} P_h \right) = \bigvee_{v \in V} P_v = \sum_{v \in V} P_v \overset{(7.4.2)}{=} \prod_{g \in G} (1 - Q_g). \qquad \square$$

8. Higher-rank graphs

We shall now apply the conclusions above to show that higher-rank graph C^*-algebras may be seen as special cases of our construction. See [11] for definitions and a detailed treatment of higher-rank graph C^*-algebras.

Before we embark on the study of k-graphs from the point of view of semigroupoids let us propose a generalization of the notion of higher-rank graphs to semigroupoids which are not necessarily categories. We will not draw any conclusions based on this notion, limiting ourselves to note that it is a natural extension of Kumjian and Pask's interesting idea.

Definition 8.1. Let k be a natural number. A rank k semigroupoid, or a k-semigroupoid, is a pair (Λ, d), where Λ is a semigroupoid and

$$d : \Lambda \to \mathbf{N}^k,$$

is a function such that

(i) for every $(f, g) \in \Lambda^{(2)}$, one has that $d(fg) = d(f) + d(g)$,
(ii) if $f \in \Lambda$, and $n, m \in \mathbf{N}^k$ are such that $d(f) = n + m$, there exists a unique pair $(g, h) \in \Lambda^{(2)}$ such that $d(g) = n$, $d(h) = m$, and $gh = f$.

For example, the Markov semigroupoid is a 1-semigroupoid, if equipped with the word length function.

Let (Λ, d) be a k-graph. In particular Λ is a category and hence a semigroupoid. Under suitable hypothesis we shall now prove that the C^*-algebra of the subjacent semigroupoid is isomorphic to the C^*-algebra of Λ, as defined by Kumjian and Pask in [11, 1.5]. In particular it will follow that the dimension function d is superfluous for the definition of the corresponding C^*-algebra.

As before, if $v \in \mathrm{obj}(\Lambda)$ we will denote by Λ^v the set of elements $f \in \Lambda$ for which $r(f) = v$. For every $n \in \mathbf{N}^k$ we will moreover let

$$\Lambda^v_n = \{ f \in \Lambda^v \colon d(f) = n \}.$$

We should observe that Λ^v_n is denoted $\Lambda^n(v)$ in [11].

According to [11, 1.4], Λ is said to *have no sources* if Λ^v_n is never empty. In case Λ^v_n is finite for every v and n one says that Λ is *row-finite*.

Notice that the absence of sources is a much more stringent condition than to require that Λ has no springs, according to Definition 3.1. In fact, since Λ is a category, and hence $s(f) \in \Lambda^f$, for every $f \in \Lambda$, we see that $\Lambda^f \neq \emptyset$, and hence higher-rank graphs automatically have no springs!

Below we will work under the standing hypotheses of [11], but we note that our construction is meaningful regardless of these requirements, so it would be interesting to compare our construction with [7] where these hypotheses are not required. This said, we suppose throughout that Λ is a k-graph for which

$$0 < |\Lambda^v_n| < \infty, \quad \forall v \in \mathrm{obj}(\Lambda),\ \forall n \in \mathbf{N}^k. \tag{8.2}$$

Lemma 8.3. *For every object v of Λ and every $n \in \mathbf{N}^k$ one has that Λ^v_n is partition of Λ^v.*

Proof. Suppose that $f, g \in \Lambda^v_n$ are such that $f \Cap g$. So there are $p, q \in \Lambda$ such that $fp = gq$. Since $d(f) = n = d(g)$ we have that $f = g$, by the uniqueness of the factorization. This shows that the elements of Λ^v_n are pairwise disjoint.

In order to show that Λ^v_n is a covering of Λ^v, let $g \in \Lambda^v$. By (8.2) pick any $h \in \Lambda^{s(g)}_n$. Since

$$d(gh) = d(g) + d(h) = d(g) + n = n + d(g),$$

we may write $gh = fk$, with $d(f) = n$, and $d(k) = d(g)$. It follows that $f \in \Lambda^v_n$ and $g \Cap f$. $\square$

For the convenience of the reader we now reproduce the definition of the C^*-algebra of a k-graph from [11, 1.5].

Definition 8.4. Given a k-graph Λ satisfying (8.2), the C^*-algebra of Λ, denoted by $C^*(\Lambda)$, is defined to be the universal C^*-algebra generated by a family $\{S_f \colon f \in \Lambda\}$ of partial isometries satisfying:

(i) $\{S_v \colon v \in \mathrm{obj}(\Lambda)\}$ is a family of mutually orthogonal projections,
(ii) $S_{fg} = S_f S_g$ for all $f, g \in \Lambda$ such that $s(f) = r(g)$,
(iii) $S^*_f S_f = S_{s(f)}$ for all $f \in \Lambda$,
(iv) for every object v and every $n \in \mathbf{N}^k$ one has $S_v = \sum_{f \in \Lambda^v_n} S_f S^*_f$.

The following is certainly well known to specialists in higher-rank graph C^*-algebras:

Proposition 8.5. *For every f and g in Λ one has that*

(i) *if $f \perp g$ then $S_f S^*_f \perp S_g S^*_g$,*
(ii) *$S_f S^*_f$ commutes with $S_g S^*_g$.*

Proof. Recall from [11, 3.1] that whenever $n \in \mathbf{N}^k$ is such that $d(f), d(g) \leqslant n$, we have

$$S_f^* S_g = \sum S_p S_q^*,$$

where the sum extends over all pair (p,q) of elements in Λ such that $fp = gq$, and $d(fp) = n$. So

$$S_f S_f^* S_g S_g^* = \sum_{p,q} S_f S_p S_q^* S_g^* = \sum_{p,q} S_{fp} S_{gq}^*.$$

Since the last expression is symmetric with respect to f and g, we see that (ii) is proved. Moreover, when $f \perp g$, it is clear that there exist no pairs (p,q) for which $fp = gq$, and hence (i) is proved as well. $\square$

We shall now prove that the crucial axiom 8.4(iv) generalizes to coverings. Observing that the notion of *exhaustive* set given in [20, 2.4] coincides with our notion of coverings, the following may also be deduced from [20, B.3].

Lemma 8.6. *Let v be an object of Λ. If H is a finite covering of Λ^v then*

$$S_v = \bigvee_{h \in H} S_h S_h^*.$$

Proof. Let $n \in \mathbf{N}^k$ with $n \geqslant d(h)$, for every $h \in H$. For all $f \in \Lambda_n^v$ we know that there is some $h \in H$ such that $f \Cap h$, so we may write $fx = hy$, for suitable x and y. Since $d(f) = n \geqslant d(h)$, we may write $f = f_1 f_2$, with $d(f_1) = d(h)$. Noticing that

$$f_1 f_2 x = hy,$$

we deduce from the unique factorization property that $f_1 = h$, which amounts to saying that $h \mid f$. We claim that this implies that $S_f S_f^* \leqslant S_h S_h^*$. In fact

$$S_h S_h^* S_f S_f^* = S_{f_1} S_{f_1}^* S_{f_1} S_{f_2} S_f^* = S_{f_1} S_{f_2} S_f^* = S_f S_f^*.$$

Summarizing, we have proved that for every $f \in \Lambda_n^v$, there exists $h \in H$, such that $S_f S_f^* \leqslant S_h S_h^*$. Therefore

$$S_v \overset{8.4(\mathrm{iv})}{=} \sum_{f \in \Lambda_n^v} S_f S_f^* = \bigvee_{f \in \Lambda_n^v} S_f S_f^* \leqslant \bigvee_{h \in H} S_h S_h^* \leqslant S_v,$$

from where the conclusion follows. $\square$

Theorem 8.7. *If Λ is a k-graph satisfying* (8.2) *then $C^*(\Lambda)$ is $*$-isomorphic to $\mathcal{O}(\Lambda)$.*

Proof. Throughout this proof we denote the standard generators of $C^*(\Lambda)$ by $\{\check{S}_f\}_{f \in \Lambda}$, together with their initial and final projections $\check{Q}_f$ and $\check{P}_f$, respectively. Meanwhile the standard generators of $\tilde{\mathcal{O}}(\Lambda)$ will be denoted by $\{\hat{S}_f\}_{f \in \Lambda}$, along with their initial and final projections $\hat{Q}_f$ and $\hat{P}_f$. In particular the $\check{S}_f$ are known to satisfy 8.4(i)–(iv), while the $\hat{S}_f$ are known to give a tight representation of the semigroupoid Λ.

Working within the semigroupoid C^*-algebra $\tilde{\mathcal{O}}(\Lambda)$, we begin by arguing that the $\hat{S}_f$ also satisfy 8.4(i)–(iv). In fact 8.4(i) follows from 7.1(i)–(ii), while 8.4(ii) is a consequence of 4.1(ii). With respect to 8.4(iii) it was proved in 7.1(iii). Finally 8.4(iv) results from the combination of 8.3 and 7.4(i).

Therefore, by the universal property of $C^*(\Lambda)$, there exists a $*$-homomorphism

$$\Phi : C^*(\Lambda) \to \tilde{\mathcal{O}}(\Lambda),$$

such that $\Phi(\check{S}_f) = \hat{S}_f$, for every $f \in \Lambda$. Evidently the range of Φ is contained in the closed $*$-subalgebra of $\tilde{\mathcal{O}}(\Lambda)$ generated by the $\hat{S}_f$, also known as $\mathcal{O}(\Lambda)$.

We next move our focus to the higher-rank graph algebra $C^*(\Lambda)$, and prove that the correspondence

$$f \in \Lambda \mapsto \check{S}_f \in C^*(\Lambda)$$

is a tight representation. Skipping the obvious 4.1(i) we notice that 4.1(ii) follows from 8.4(ii) when $s(f) = r(g)$. On the other hand, if $s(f) \neq r(g)$ we have

$$\check{S}_f \check{S}_g = \check{S}_f \check{S}_{s(f)} \check{S}_{r(g)} \check{S}_g = 0,$$

by 8.4(i).

We next claim that the initial and final projections of the $\check{S}_f$ commute among themselves. That two initial projections commute follows from 8.4(i) and (iii). Speaking of the commutativity between an initial projection $\check{Q}_f$ and a final projection $\check{P}_g$, we have that $\check{Q}_f = \check{S}_{s(f)}$, by 8.4(iii) and $P_g \leqslant S_{r(g)}$ by 8.4(iv). So either $\check{P}_g \leqslant \check{Q}_f$, if $s(f) = r(g)$, or $\check{P}_g \perp \check{Q}_f$, if $s(f) \neq r(g)$, by 8.4(i). In any case it is clear that $\check{Q}_f$ and $\check{P}_g$ commute. That two final projections commute is precisely the content of 8.5(ii).

Clearly 4.1(iii) is granted by 8.5(i). In order to prove 4.1(iv) let $f, g \in \Lambda$ with $s(f) = r(g)$. We then have that

$$\check{Q}_f \check{P}_g = \check{S}_f^* \check{S}_f \check{P}_g = \check{S}_{s(f)} \check{P}_g = \check{S}_{r(g)} \check{S}_g \check{S}_g^* = \check{S}_g \check{S}_g^* = \check{P}_g.$$

This shows that $\check{S}$ is a representation of Λ in $C^*(\Lambda)$, which we will now prove to be tight. For this let

$$\pi : C^*(\Lambda) \to \mathcal{B}(H)$$

be a faithful nondegenerate representation of $C^*(\Lambda)$. Through π we will view $C^*(\Lambda)$ as a subalgebra of $\mathcal{B}(H)$, and hence we may consider $\check{S}$ as a representation of Λ on H. It is clear that $\check{S}$ is nondegenerate, according to Definition 7.2. By 7.4(ii) it is then enough to show that for every $v \in \operatorname{obj}(\Lambda)$ and every finite partition H of v one has that

$$\bigvee_{h \in H} \check{P}_h = \check{P}_v,$$

but this is precisely what was proved in 8.6.

By the universal property of $\tilde{\mathcal{O}}(\Lambda)$ there is a $*$-homomorphism

$$\Psi : \tilde{\mathcal{O}}(\Lambda) \to B(H)$$

such that $\Psi(\hat{S}_f) = \check{S}_f$, for every $f \in \Lambda$. Clearly $\Psi(\mathcal{O}(\Lambda)) \subseteq C^*(\Lambda)$, so we may then view Ψ and Φ as maps

$$\Psi : \mathcal{O}(\Lambda) \to C^*(\Lambda) \quad \text{and} \quad \Phi : C^*(\Lambda) \to \mathcal{O}(\Lambda),$$

which are obviously each others inverses. $\square$

References

[1] T. Bates, J. Hong, I. Raeburn, W. Szymański, The ideal structure of the C^*-algebras of infinite graphs, Illinois J. Math. 46 (2002) 1159–1176.
[2] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C^*-algebras of row-finite graphs, New York J. Math. 6 (2000) 307–324 (electronic).
[3] B. Blackadar, Shape theory for C^*-algebras, Math. Scand. 56 (1985) 249–275.
[4] R. Exel, Partial actions of groups and actions of inverse semigroups, Proc. Amer. Math. Soc. 126 (1998) 3481–3494.
[5] R. Exel, Inverse semigroups and combinatorial C^*-algebras, Bull. Braz. Math. Soc. (N.S.) 39 (2008) 191–313.
[6] R. Exel, M. Laca, Cuntz–Krieger algebras for infinite matrices, J. Reine Angew. Math. 512 (1999) 119–172.
[7] C. Farthing, P. Muhly, T. Yeend, Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach, Semigroup Forum 71 (2005) 159–187.
[8] N. Fowler, M. Laca, I. Raeburn, The C^*-algebras of infinite graphs, Proc. Amer. Math. Soc. 128 (2000) 2319–2327.
[9] T. Katsura, A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results, Trans. Amer. Math. Soc. 356 (2004) 4287–4322 (electronic).
[10] A. Kumjian, D. Pask, C^*-algebras of directed graphs and group actions, Ergodic Theory Dynam. Systems 19 (1999) 1503–1519.
[11] A. Kumjian, D. Pask, Higher-rank graph C^*-algebras, New York J. Math. 6 (2000) 1–20 (electronic).
[12] A. Kumjian, D. Pask, I. Raeburn, Cuntz–Krieger algebras of directed graphs, Pacific J. Math. 184 (1998) 161–174.
[13] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids, and Cuntz–Krieger algebras, J. Funct. Anal. 144 (1997) 505–541.
[14] D. Pask, J. Quigg, I. Raeburn, Fundamental groupoids of k-graphs, New York J. Math. 10 (2004) 195–207 (electronic).
[15] D. Pask, I. Raeburn, M. Rordam, A. Sims, Rank-two graphs whose C^*-algebras are direct limits of circle algebras, J. Funct. Anal. 239 (2006) 137–178.
[16] A.L.T. Paterson, Graph inverse semigroups, groupoids and their C^*-algebras, J. Operator Theory 48 (2002) 645–662.
[17] J. Renault, A Groupoid Approach to C^*-Algebras, Lecture Notes in Math., vol. 793, Springer, 1980.
[18] I. Raeburn, Graph Algebras, CBMS Reg. Conf. Ser. Math., vol. 103, 2005, vi+113 pp.
[19] I. Raeburn, A. Sims, T. Yeend, Higher-rank graphs and their C^*-algebras, Proc. Edinb. Math. Soc. 46 (2003) 99–115.
[20] I. Raeburn, A. Sims, T. Yeend, The C^*-algebras of finitely aligned higher-rank graphs, J. Funct. Anal. 213 (2004) 206–240.
[21] I. Raeburn, W. Szymański, Cuntz–Krieger algebras of infinite graphs and matrices, Trans. Amer. Math. Soc. 356 (2004) 39–59 (electronic).
[22] I. Raeburn, M. Tomforde, D.P. Williams, Classification theorems for the C^*-algebras of graphs with sinks, Bull. Aust. Math. Soc. 70 (2004) 143–161.
[23] G. Robertson, T. Steger, C^*-algebras arising from group actions on the boundary of a triangle building, Proc. Lond. Math. Soc. 72 (1996) 613–637.
[24] G. Robertson, T. Steger, Affine buildings, tiling systems and higher-rank Cuntz–Krieger algebras, J. Reine Angew. Math. 513 (1999) 115–144.
[25] M. Tomforde, A unified approach to Exel–Laca algebras and C^*-algebras associated to graphs, J. Operator Theory 50 (2003) 345–368.
[26] Y. Watatani, Graph theory for C^*-algebras, in: Operator Algebras and Applications, Part I, Kingston, Ont., 1980, in: Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, RI, 1982, pp. 195–197.
[27] T. Yeend, Groupoid models for the C^*-algebras of topological higher-rank graphs, J. Operator Theory 57 (1) (2007) 95–120.